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Coherent States and Number–Phase Uncertainty Relations

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The number–phase uncertainty relations are revisited in view of the recent discovery of a proper covariant phase observable. The high-amplitude limits of the coherent-state expectations of the moment operators of the phase observable are determined and the behavior of the number–phase uncertainty product in that limit is investigated.

1. INTRODUCTION

It is well known that there is no phase observable, given as a self-adjoint operator, which would be covariant under the shifts generated by the number observable (Carruthers and Nieto, 1968; Garrison and Wong, 1970; Lévy-Leblond, 1976). It is equally well known that there are self-adjoint operators Φ which are conjugate to the number N in the sense of the commutation relation $\Phi N - N\Phi = iI$ (valid on a dense domain) (Garrison and Wong, 1970; Galindo, 1984; Busch *et al.*, 1995a). It is another matter of fact that there are phase observables, given as semi-spectral measures, which are phase-shift-covariant and whose first moment operators fulfil the Heisenberg commutation relation with the number observable (Holevo, 1982; Busch *et al.*, 1995b). In this paper we shall examine the uncertainty relations of the number observable and a particular covariant phase observable, which is uniquely given by the polar decomposition of the annihilation operator associated with the number (Section 2). We study the coherent state expectations of the moment operators and the noise operator of the phase observable in the limit of large amplitude (Section 4) and we determine the behavior of the number–phase uncertainty relations in that limit (Section 5).

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2. THE PHASE

Let $N = \Sigma_{n=0}^{\infty} n|n\rangle\langle n|$ be the number observable, with the domain $D(N) = \{\psi \in \mathcal{H} | \Sigma_{n=0}^{\infty} n^2 |\langle n|\psi\rangle|^2 < \infty\}$ on a complex separable Hilbert space $\mathcal{H}$. Let $a = \Sigma_{n=0}^{\infty} \sqrt{n+1}|n\rangle\langle n+1|$ and $a^* = \Sigma_{n=0}^{\infty} \sqrt{n+1}|n+1\rangle\langle n|$ be the annihilation and creation operators associated with $N = a^*a$, where, for instance, $|n\rangle\langle n+1|$ denotes the operator $\psi \mapsto |n\rangle\langle n+1|\psi := \langle n+1|\psi\rangle|n\rangle$ on $\mathcal{H}$. Let $a = V|a| = V\sqrt{N}$ be the polar decomposition of a. The partial isometry $V = \Sigma_{n=0}^{\infty}|n\rangle\langle n+1|$ is not unitary, but it is contractive. Hence there is a *unique* semi-spectral measure M such that $V^k = \int_0^{2\pi} e^{ikx} M(dx)$, for all $k = 0, \pm 1, \pm 2, \ldots$ (Mlak, 1991). The structure of this measure is easily determined, and one obtains for all Borel subsets X of the interval $[0, 2\pi]$

$$M(X) = \sum_{n,m=0}^{\infty} \frac{1}{2\pi} \int_X e^{i(n-m)x} dx\, |n\rangle\langle m| \tag{1}$$

A direct computation shows that M is *covariant* under the phase shifts generated by the number: $e^{ixN} M(X) e^{-ixN} = M(X + x)$ for all $x \in [0, 2\pi]$ and $X \in \mathcal{B}[0, 2\pi]$.

The *moment operators* of M,

$$M^{(k)} := \int_0^{2\pi} x^k M(dx) \tag{2}$$

are bounded, self-adjoint operators on $\mathcal{H}$ for all $k = 0, 1, 2, \ldots$. The first moment operator $M^{(1)}$ assumes the form

$$M^{(1)} = \sum_{n \neq m} \frac{1}{i(n-m)} |n\rangle\langle m| + \pi I \tag{3}$$

and it is just the 'phase operator' studied by Garrison and Wong (1970) and Galindo (1984). As shown by those authors, the operator $M^{(1)}$ fulfils the commutation relation

$$M^{(1)}N - NM^{(1)} = iI \tag{4}$$

on the domain $\{\psi \in \mathcal{H} | M^{(1)} \psi \in D(N)\}$, which is dense in $\mathcal{H}$. However, since M is no spectral measure, and thus not multiplicative, its first moment operator does not yield the k^{th} moment operator $M^{(k)}$ as the k^{th} power of $M^{(1)}$. In particular, the *noise operator* $R := M^{(2)} - (M^{(1)})^2$ is strictly positive (Riesz and Sz.-Nagy, 1990). Thus, in order to get the whole measurement statistics of the phase observable one needs the statistics of all the moment operators of M. [For additional details of this subject matter we refer to Busch *et al.*

(1994) and further references therein.] Especially, the variance of the phase observable M can now be given as

$$\begin{aligned}\mathrm{Var}(M, \psi) &:= \int_0^{2\pi} x^2 \langle\psi|M(dx)\psi\rangle - \left(\int_0^{2\pi} x\langle\psi|M(dx)\psi\rangle\right)^2 \\ &= \langle\psi|M^{(2)}\psi\rangle - \langle\psi|(M^{(1)})^2\psi\rangle \\ &\quad + \langle\psi|(M^{(1)})^2\psi\rangle - \langle\psi|M^{(1)}\psi\rangle^2 \\ &= \langle\psi|R\psi\rangle + \mathrm{Var}(M^{(1)}, \psi)\end{aligned} \tag{5}$$

This shows that for any vector state $\psi \in \mathcal{H}$, $0 \leq \mathrm{Var}(M, \psi) \leq (2\pi)^2$. Before studying the behavior of the number–phase uncertainty product Var (N, ψ) Var (M, ψ) we determine the number state and the coherent state expectations of the moment operators $M^{(k)}$ of the phase observable M.

3. NUMBER STATES AND THE PHASE

The number state expectations of $M^{(k)}$,

$$\langle n|M^{(k)}|n\rangle = \int_0^{2\pi} x^k \langle n|M(dx)|n\rangle \tag{6}$$

are easily computed, since the density $\langle n|M(dx)|n\rangle$ is now simply $(1/2\pi)dx$. One gets $\langle n|M^{(k)}|n\rangle = (2\pi)^k/(k + 1)$, so that, in particular, Var $(M,|n\rangle) = \frac{1}{3}\pi^2$. This exhibits the randomness of the phase in the number states. We observe also that $\langle n|R|n\rangle = \pi^2/6 - \Sigma_1^n\,(1/k^2)$, and Var $(M^{(1)},|n\rangle) = \pi^2/6 + \Sigma_1^n\,(1/k^2)$. Thus, with growing n, the noise expectation $\langle n|R|n\rangle$ tends to zero and the quantity Var $(M^{(1)},|n\rangle)$ grows to Var$(M,|n\rangle)$.

4. COHERENT STATES AND THE PHASE

It is well known that coherent states have a well-defined phase in the limit of large amplitude (see, e.g., Walls and Milburn, 1994). We shall show next that the phase observable M behaves accordingly. Therefore, consider the expectation of the k^{th} moment operator $M^{(k)}$ of M in a coherent state $|z\rangle = exp\,(-|z|^2/2)\,\Sigma_{n=0}^{\infty}\,(z^n/\sqrt{n!})|n\rangle$,

$$\langle z|M^{(k)}|z\rangle = \int_0^{2\pi} x^k \langle z|M(dx)|z\rangle \tag{7}$$

To calculate this expectation, we determine first the density $\langle z|M(dx)|z\rangle$. Let $z = |z|e^{i\theta} = re^{i\theta}$ be the polar decomposition of the complex number $z \in \mathbf{C}$, and define the complex function $g(w) := \Sigma_{n=0}^{\infty} w^n/\sqrt{n!}$. The density can then be given as follows:

$$\langle z|M(dx)|z\rangle = \sum_{n,m=0}^{\infty} \langle z|n\rangle\langle m|z\rangle e^{i(n-m)x}\,\frac{dx}{2\pi}$$

$$= e^{-r^2} g(re^{-i(\theta-x)})g(re^{i(\theta-x)})\,\frac{dx}{2\pi} \tag{8}$$

The expectation (7) thus gets the form

$$\langle z|M^{(k)}|z\rangle = e^{-r^2}\int_0^{2\pi} x^k g(re^{-i(\theta-x)})g(re^{i(\theta-x)})\,\frac{dx}{2\pi} \tag{9}$$

To determine this integral for large amplitudes r we use the fact that the function g behaves asymptotically (for large r, with $w = re^{i\alpha}$) as follows (Garrison and Wong, 1970):

$$g(re^{i\alpha}) \to (2\pi)^{1/4}(2r)^{1/2}\exp\left[\frac{1}{2}r^2 - r^2\alpha^2 + i\left(r^2 - \frac{1}{2}\right)\alpha\right] \tag{10}$$

Introducing a new variable $u = \sqrt{2}r(x - \theta)$, we obtain for the quantity (9) the asymptotic form

$$\langle z|M^{(k)}|z\rangle \to \frac{1}{\sqrt{\pi}}\sum_{l=0}^{k}\binom{k}{l}\theta^{k-l}(\sqrt{2}r)^{-l}\int_{-\sqrt{2}r\theta}^{\sqrt{2}r(2\pi-\theta)} u^l e^{-u^2}\,du \tag{11}$$

In the limit $r \to \infty$ the above integral develops a singularity at $\theta = 0, 2\pi$. Therefore, in the subsequent discussion we assume that $0 \neq \theta \neq 2\pi$. Using the estimate

$$\left|\int_{-\sqrt{2}r\theta}^{\sqrt{2}r(2\pi-\theta)} u^l e^{-u^2}\,du\right| \leq \begin{cases} \dfrac{(2p-1)!!}{2p}\sqrt{\pi} & \text{for } l = 2p \\ p! & \text{for } l = 2p+1 \end{cases} \tag{12}$$

one gets

$$\lim_{r\to\infty}\langle z|M^{(k)}|z\rangle = \theta^k \tag{13}$$

for all $k = 0, 1, 2, \ldots$.

This result shows, first of all, that

$$\lim_{|z|\to\infty} \mathrm{Var}(M, |z\rangle) = \theta^2 - \theta^2 = 0 \tag{14}$$

Since, according to equation (5), $\mathrm{Var}(M,|z\rangle) = \langle z|R|z\rangle + \mathrm{Var}(M^{(1)},|z\rangle)$, we also have that both the expectation of the (positive) noise operator R and the variance of the first moment operator $M^{(1)}$ in a coherent state $|z\rangle$ tend to zero with increasing amplitude $|z|$. The latter result was obtained also by Garrison and Wong (1970) and Galindo (1984). As another remark on the result (13) we observe that in the limit $|z| \to \infty$ the characteristic function $\xi(t)$ of the probability distribution $X \mapsto \langle z|M(X)|z\rangle$ obtains the form $\xi(t) = e^{it\theta}$. Since $\xi(t) \neq 0$ for all t, the limiting distribution $X \mapsto \langle z|M(X)|z\rangle_{|z|\to\infty}$ has no density. Since $e^{it\theta} = (e^{it\theta/n})^n$, the distribution is infinitely divisible, a matter of fact reflecting the infinite divisibily of the phase.

To close this section we show directly that $\lim_{|z|\to\infty} \langle z|R|z\rangle = 0$. Using the fact that the semi-spectral measure M is the Neumark projection of the spectral measure of the position Q of an object confined to move in the interval $[0, 2\pi]$ onto the subspace of the positive eigenvalues of the (conjugate) momentum P, one can easily determine the explicit form of the noise operator to be as follows:

$$R = \sum_{n,m=0}^{\infty} s_{n,m}|n\rangle\langle m| \tag{15}$$

with

$$s_{n,m} = \sum_{k=1}^{\infty} \frac{1}{(n+k)(m+k)} \tag{16}$$

This gives

$$\begin{aligned} \langle z|R|z\rangle &= \sum_{n,m=0}^{\infty} s_{n,m}\langle z|n\rangle\langle m|z\rangle \\ &= \sum_{n=0}^{\infty} s_{n,m}|\langle n|z\rangle|^2 + \sum_{n\neq m} s_{n,m}\langle z|n\rangle\langle m|z\rangle \equiv \mathrm{I} + \mathrm{II} \end{aligned} \tag{17}$$

To estimate the second term we use the fact that, for instance, for $n > m$ one has

$$s_{n,m} = \frac{1}{n-m} \sum_{k=0}^{n-m-1} \frac{1}{m+1+k}$$

Hence we have $1/(n+1) < s_{n,m} < 1/(m+1)$, so that $s_{n,m}$ lies in the interval $(1/(n+1), 1/(m+1))$. We can approximate $s_{n,m}$ by the geometric mean of the endpoints of this interval:

$$s_{n,m} \simeq \frac{1}{\sqrt{n+1}\,\sqrt{m+1}} =: f_{n,m}$$

This shows that for $n > m$, $s_{n,m}$ is asymptotically equivalent to $f_{n,m}$ in the sense that $\lim_{n,m\to\infty} s_{n,m}/f_{n,m} = 1$ (Murray, 1984). We obtain the same approximation for $n < m$, so that for any $n \neq m$ we have $s_{n,m} \simeq f_{n,m}$. Thus the second term in (17) takes the form

$$\begin{aligned} \mathrm{II} &\simeq \sum_{n \neq m} f_{n,m} \langle z|n\rangle\langle m|z\rangle \\ &= \sum_{n,m=0}^{\infty} f_{n,m} \langle z|n\rangle\langle m|z\rangle - \sum_{n=0}^{\infty} f_{n,n} \langle z|n\rangle\langle n|z\rangle \\ &= \frac{e^{-|z|^2}}{|z|^2} (g(\bar{z}) - 1)(g(z) - 1) - \frac{1}{|z|^2}(1 - e^{-|z|^2}) \end{aligned} \tag{18}$$

which shows, by equation (10), that $|\mathrm{II}| \to 0$, whenever $|z| \to \infty$. To estimate the first series in (17) we observe that

$$s_{n,m} \leq \int_0^\infty \frac{dx}{(n + x)^2} = \frac{1}{n}$$

Therefore,

$$I \leq e^{-|z|^2}\left(1 + \sum_{n=1}^{\infty} \frac{|z|^2}{n!n}\right) = e^{-|z|^2}(1 + Ei(|z|^2) - \ln |z|^2 - C) \tag{19}$$

Since

$$e^{-|z|^2} Ei(|z|^2) = |z|^{-2} \sum_0^\infty \frac{n!}{|z|^{2n}} \to 0 \qquad \text{with} \quad |z| \to \infty$$

the above estimate gives $\lim_{|z|\to\infty} I = 0$. Putting these two estimates together, one obtains

$$|\langle z|R|z\rangle| \leq \mathrm{I} + |\mathrm{II}| \to 0 \qquad \text{with} \quad |z| \to \infty \tag{20}$$

5. NUMBER–PHASE UNCERTAINTY

The number observable N is unbounded. The variance of N in a vector state ψ is defined (and is finite) if and only if $\psi \in D(N)$. The phase observable M is bounded, so that its variance is finite for all vector states ψ; in fact, $\mathrm{Var}(M, \psi) \leq (2\pi)^2$. Thus for any vector state $\psi \in D(N)$, the uncertainty product $\mathrm{Var}(N, \psi)\mathrm{Var}(M, \psi) \geq 0$ is well defined, and this product is equal to 0 for the number eigenstates $|n\rangle$. Apart from that, there is kind of uncertainty relation for the number–phase pair (N, M), which we formulate next.

By equation (5) the variance of M in any state ψ is the sum of two terms, the (nonnegative) noise $\langle\psi|R\psi\rangle$ and the variance $\mathrm{Var}(M^{(1)}, \psi)$ of the first moment operator in that state. Hence we have for all vector states $\psi \in D(N)$

$$\begin{aligned}\mathrm{Var}(N, \psi)\, \mathrm{Var}(M, \psi) &= \mathrm{Var}(N, \psi)\langle\psi|R\psi\rangle + \mathrm{Var}(N, \psi)\, \mathrm{Var}(M^{(1)}, \psi)\\ &\geq \mathrm{Var}(N, \psi)\, \mathrm{Var}(M^{(1)}, \psi)\\ &\geq \frac{1}{4}\left|\langle N\psi|M^{(1)}\psi\rangle - \langle M^{(1)}\psi|N\psi\rangle\right|^2 \end{aligned} \tag{21}$$

Due to the commutation relation (4) one has, in addition, the inequality

$$\mathrm{Var}(N, \psi)\, \mathrm{Var}(M, \psi) \geq \frac{1}{4} \tag{22}$$

for any vector state ψ for which $M^{(1)}\psi \in D(N)$.

We study next the behavior of the number–phase uncertainty product in the coherent states $|z\rangle$, $z \in \mathbf{C}$. Though $|z\rangle \in D(N)$, these states are not in the domain of the commutator $NM^{(1)} - M^{(1)}N$. Anyway, it turns out that in the high amplitude the uncertainty product of the number and the phase observables approaches, in general, the value $\frac{1}{4}$.

Consider a coherent state $|z\rangle$, and let $z = re^{i\theta}$. The variance of the number observable is simply $\mathrm{Var}(N,|z\rangle) = |z|^2$, whereas $\mathrm{Var}(M,|z\rangle)$ is to be determined from equation (5). To estimate this quantity for large r we denote

$$I_l(r, \theta) := \int_{-\sqrt{2}r\theta}^{\sqrt{2}r(2\pi-\theta)} u^l e^{-u^2}\, du \tag{23}$$

and use the estimates (11) to get

$$\begin{aligned}\mathrm{Var}(M, |z\rangle) &= \frac{\theta^2}{\sqrt{\pi}} I_0(r, \theta)\left(1 - \frac{1}{\sqrt{\pi}} I_0(r, \theta)\right)\\ &\quad + \frac{1}{r}\sqrt{\frac{2}{\pi}}\, \theta I_1(r, \theta)\left(1 - \frac{1}{\sqrt{\pi}} I_0(r, \theta)\right)\\ &\quad + \frac{1}{2r^2\sqrt{\pi}} I_2(r, \theta) - \frac{1}{2\pi r^2} I_1(r, \theta)^2 \end{aligned} \tag{24}$$

Assume now that $0 \neq \theta \neq 2\pi$. Using the rule of L'Hôspital, one can show by direct computation that the first two terms in (23), when multiplied by r^2, tend separately to zero with $r \to \infty$. Therefore,

$$\begin{aligned}\lim_{r\to\infty} \mathrm{Var}(N, |z\rangle)\mathrm{Var}(M, |z\rangle) &= \lim_{r\to\infty} \frac{1}{2\sqrt{\pi}} I_2(r, \theta) - \lim_{r\to\infty} \frac{1}{2\pi} I_1(r, \theta)^2\\ &= \frac{1}{\sqrt{\pi}} \int_0^\infty u^2 e^{-u^2}\, du = \frac{1}{4} \end{aligned} \tag{25}$$

Since $\lim_{r\to\infty}\langle z|R|z\rangle = 0$, we also have

$$\lim_{r\to\infty} \mathrm{Var}(N, |z\rangle)\mathrm{Var}(M, |z\rangle) = \lim_{r\to\infty} \mathrm{Var}(N, |z\rangle)\mathrm{Var}(M^{(1)}, |z\rangle) = \frac{1}{4} \tag{26}$$

We recall that Garrison and Wong (1970) showed the weaker result that 1/4 is a lower bound for the product $\mathrm{Var}(N,|z\rangle)\mathrm{Var}(M^{(1)},|z\rangle)$ in the limit $r \to \infty$.

Consider now a coherent state $|z\rangle$, with $\theta = 0$. Using equation (9) and the estimate (10), one can again compute the uncertainty product for N and M, and one gets

$$\begin{aligned}\lim_{r\to\infty} \mathrm{Var}(N, |z\rangle)\mathrm{Var}(M, |z\rangle) &= \frac{1}{2\sqrt{\pi}} \int_0^\infty u^2 e^{-u^2}\, du - \frac{1}{2\pi}\left(\int_0^\infty u e^{-u^2}\, du\right)^2 \\ &= \frac{1}{8}\left(1 - \frac{1}{\pi}\right)\end{aligned} \tag{27}$$

This number is less than 0.25. The same result is obtained for $\theta = 2\pi$. This shows that the uncertainty product of the number and phase observables has a discontinuity at $z = \infty + i0$; namely, we have

$$\lim_{\theta\to 0}\lim_{r\to\infty} \mathrm{Var}(N, |z\rangle)\mathrm{Var}(M, |z\rangle) \neq \lim_{r\to\infty}\lim_{\theta\to 0} \mathrm{Var}(N, |z\rangle)\mathrm{Var}(M, |z\rangle) \tag{28}$$

Hence in the complex plane the limit of this product does not exist for $z \to \infty + i0$.

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